ESTABILIDAD III

Clase 16: Introducción al Análisis Dinámico

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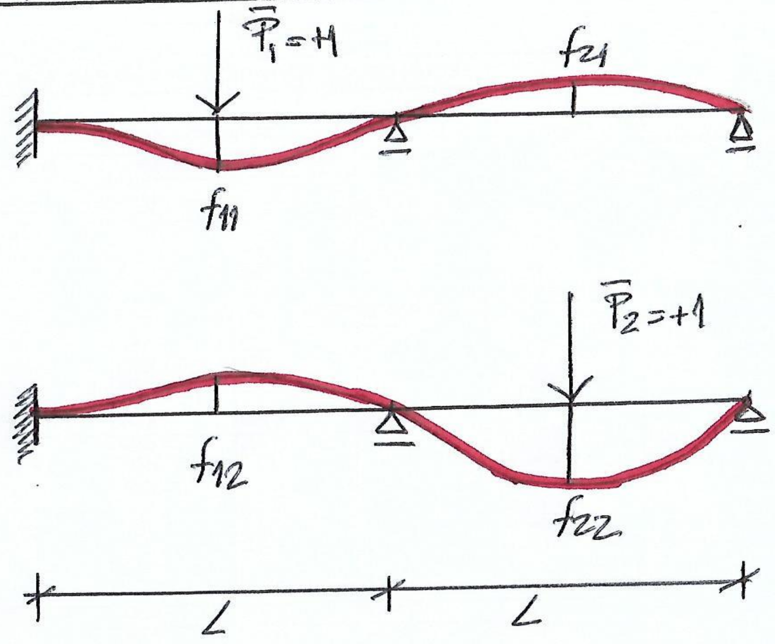
18/06/2020

ESTABILIDAD III - CURSO 1 - CLASE 16 - 18/06/2020 – Introducción al Análisis Dinámico

16.1 EJ. MATRIZ DE RIGIDEZ CONDENSADA - Obtenida a partir de una Matriz de Flexibilidad



$L = 5\ m$

$EJ = 32000\ kNm^2$

$$\underline{F} = 10^{-8} \begin{bmatrix} 2{,}906 & -1{,}308 \\ -1{,}308 & 5{,}522 \end{bmatrix} \left[\frac{m}{N}\right]$$

$$\underline{R} = \underline{F}^{-1} = 10^{8} \begin{bmatrix} 0{,}3852 & 0{,}0912 \\ 0{,}0912 & 0{,}2027 \end{bmatrix} \left[\frac{N}{m}\right]$$

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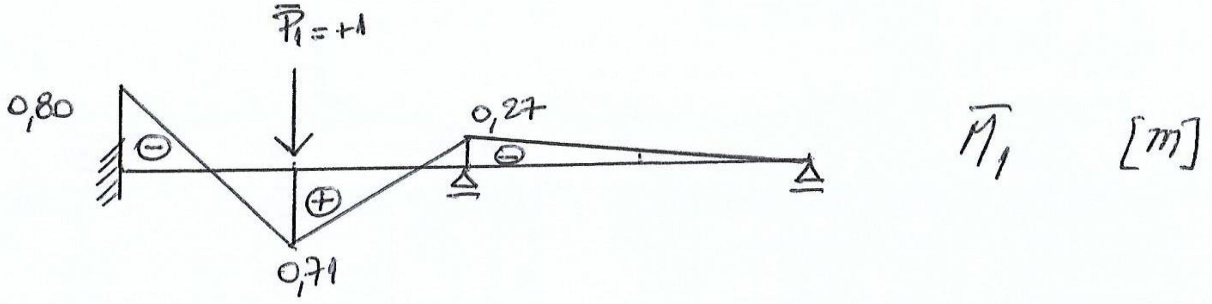
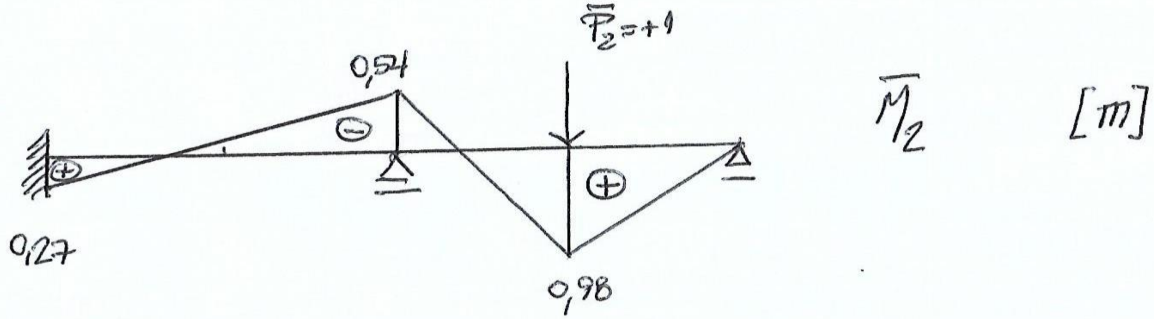
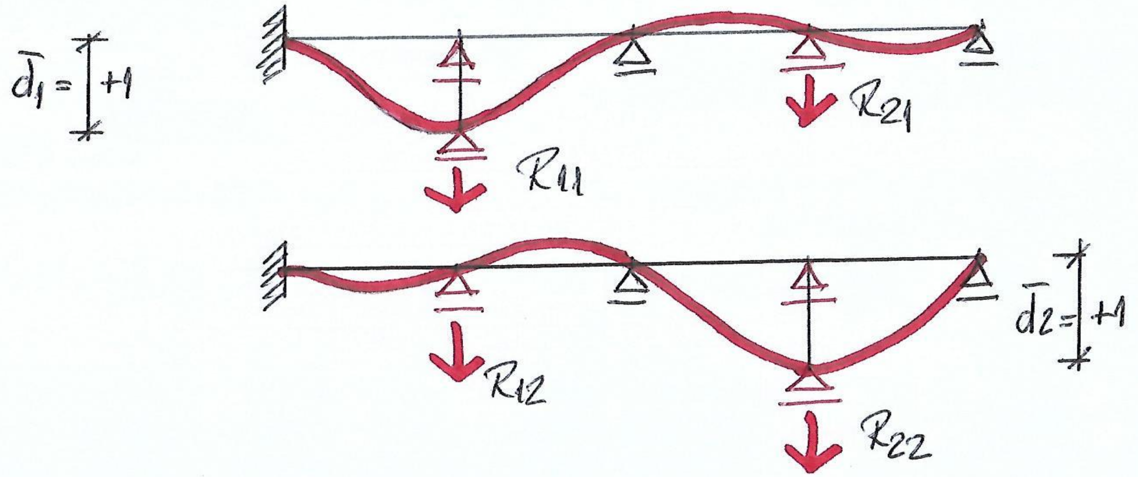




16.2 MATRIZ DE RIGIDEZ CONDENSADA – Obtenida con cedimientos de vínculo unitarios en sistemas indeterminados



$$\underline{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = 10^8 \begin{bmatrix} 0{,}3851 & 0{,}0912 \\ 0{,}0912 & 0{,}2027 \end{bmatrix} \quad \left[\frac{N}{m}\right]$$

16.3 - EJEMPLO DE CÁLCULO DE PULSACIONES NATURALES Y MODOS PROPIOS DE VIBRACIÓN

$M_1 = M_2 = 1000\ kg$

$$\underline{M} = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \qquad \underline{M} = 10^3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad [kg]$$

ECUACIÓN CARACTERÍSTICA :

$$\left| \underline{R} - \omega^2 \underline{M} \right| = 0$$

$$\left| \underline{R} - \lambda \underline{M} \right| = 0 \qquad \text{con} \quad \lambda = \omega^2$$

$$\left| \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} - \lambda \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \right| = 0$$

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$$\begin{vmatrix} (R_{11} - \lambda M_1) & R_{12} \\ R_{21} & (R_{22} - \lambda M_2) \end{vmatrix} = 0$$

$$10^3 \begin{bmatrix} (38520 - \lambda) & 9120 \\ 9120 & (20270 - \lambda) \end{bmatrix} = 0$$

$$(38520 - \lambda) \cdot (20270 - \lambda) - (9120)^2 = 0$$

$$\lambda^2 - 58790\lambda + 697{,}626 \times 10^6 = 0$$

$$\lambda_1 = 16493{,}84 \rightarrow \omega_1 = 128{,}43 \ \frac{rad}{s}$$

$$\lambda_2 = 42296{,}16 \rightarrow \omega_2 = 205{,}66 \ \frac{rad}{s}$$

$$\begin{bmatrix} (R_{11}-\lambda_1 . M_1) & R_{12} \\ R_{21} & (R_{22}-\lambda_1 M_2) \end{bmatrix} \cdot \begin{Bmatrix} V_1^1 \\ V_2^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Si $V_1^1 = 1 \Rightarrow V_2^1 = \dfrac{-(R_{11}-\lambda_1 M_1)}{R_{12}} = \dfrac{-(38520-16493{,}84)}{9120} = -2{,}415 \Rightarrow \underset{\sim}{V}^1 = \begin{Bmatrix} 1 \\ -2{,}415 \end{Bmatrix}$

$$\begin{bmatrix} (R_{11}-\lambda_2 M_1) & R_{12} \\ R_{21} & (R_{22}-\lambda_2 M_2) \end{bmatrix} \begin{Bmatrix} V_1^2 \\ V_2^2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Si $V_1^2 = 1 \Rightarrow V_2^2 = \dfrac{-(R_{11}-\lambda_2 M_1)}{R_{12}} = \dfrac{-(38520-42296{,}16)}{9120} = +0{,}414 \Rightarrow \underset{\sim}{V}^2 = \begin{Bmatrix} 1 \\ +0{,}414 \end{Bmatrix}$

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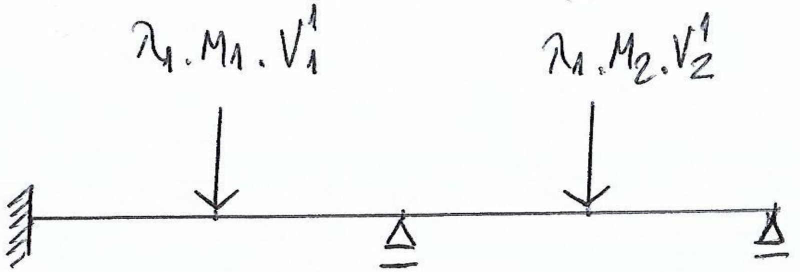

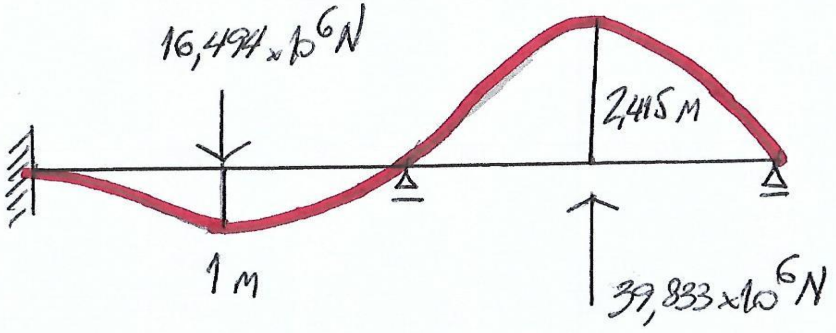

1º MODO DE VIBRACIÓN

$$\underline{R} \cdot \underset{\sim}{V}^1 = \omega_1^2 \, \underline{M} \, \underset{\sim}{V}^1$$

$$\lambda_1 \, \underline{M} \, \underset{\sim}{V}^1$$

$\frac{1}{s^2}$ Kg m

$$\lambda_1 . M_1 . V_1^1 = 16493,84 . 1000 . 1$$
$$= 16,494 . 10^6 \, N$$

$$\lambda_1 . M_2 . V_2^1 = -16493,84 . 1000 . 2,415$$
$$= -39,833 . 10^6 \, N$$

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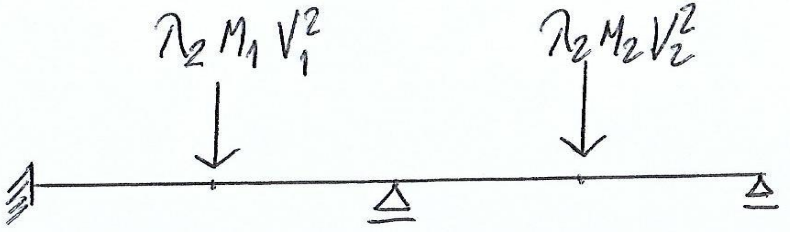
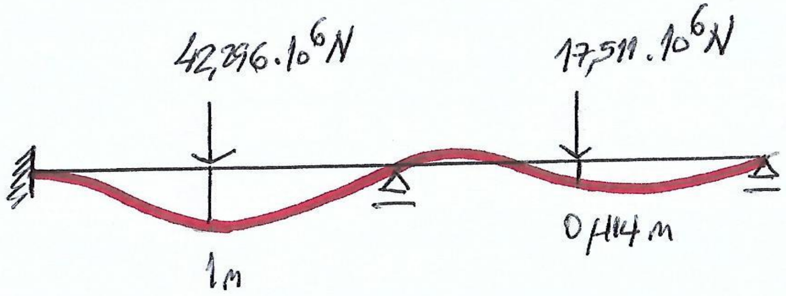


$$\underline{K} \cdot \underset{\sim}{V}^2 = \omega_2^2 \, \underline{M} \, \underset{\sim}{V}^2$$

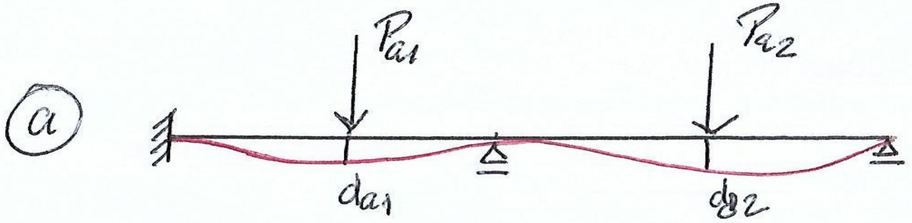


2° MODO DE VIBRACIÓN

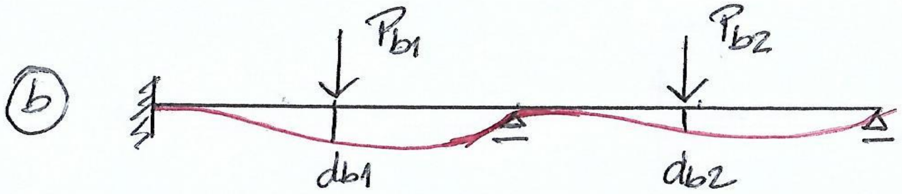
$$\lambda_2 \cdot M_1 \cdot V_1^2 = 42296{,}16 \cdot 1000 \cdot 1$$
$$= 42{,}296 \cdot 10^6\,N$$

$$\lambda_2 \cdot M_2 \cdot V_2^2 = 42296{,}16 \cdot 1000 \cdot 0{,}414$$
$$= 17{,}511 \cdot 10^6\,N$$

16.4 RECIPROCIDAD DE LOS TRABAJOS DE 2 SISTEMAS DE FUERZAS



$$\underset{\sim}{P_a} = \begin{Bmatrix} P_{a1} \\ P_{a2} \end{Bmatrix} \quad \text{provoca} \quad \underset{\sim}{d_a} = \begin{Bmatrix} d_{a1} \\ d_{a2} \end{Bmatrix}$$



$$\underset{\sim}{P_b} = \begin{Bmatrix} P_{b1} \\ P_{b2} \end{Bmatrix} \quad \text{provoca} \quad \underset{\sim}{d_b} = \begin{Bmatrix} d_{b1} \\ d_{b2} \end{Bmatrix}$$

$\downarrow \check{X}_1$ $\downarrow \check{X}_2$

$$\boxed{W_{ab} = W_{ba}}$$

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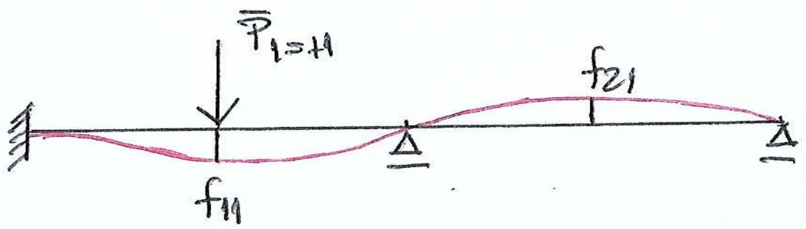
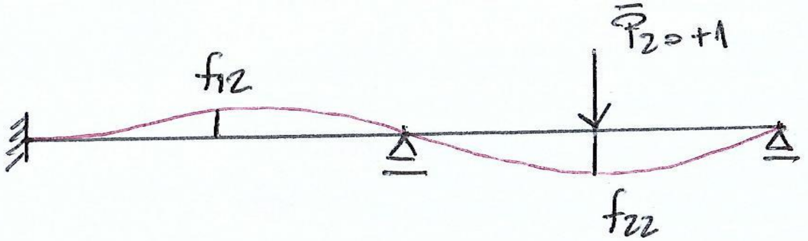


$$\underline{F} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$



$$\underset{\sim}{d_a} = \underline{F} \cdot \underset{\sim}{P_a}$$

$$\begin{cases} d_{a1} = f_{11} \cdot P_{a1} + f_{12} \cdot P_{a2} \\ d_{a2} = f_{21} \cdot P_{a1} + f_{22} \cdot P_{a2} \end{cases}$$

$$\underset{\sim}{d_b} = \underline{F} \cdot \underset{\sim}{P_b}$$

W_{ab} = Trabajo de las fuerzas del sistema ⓐ sobre la deformación que provoca el sistema ⓑ

W_{ba} = Trabajo de las fuerzas del sistema ⓑ sobre la deformación que provoca el sistema ⓐ

$$W_{ab} = P_{a1} \cdot d_{b1} + P_{a2} \cdot d_{b2} = \underset{\sim}{P_a}^{t} \cdot \underset{\sim}{d_b} = \underset{\sim}{P_a}^{t} \cdot \underline{F} \cdot \underset{\sim}{P_b}$$

$$\begin{bmatrix} P_{a1} & P_{a2} \end{bmatrix} \cdot \begin{bmatrix} d_{b1} \\ d_{b2} \end{bmatrix}$$

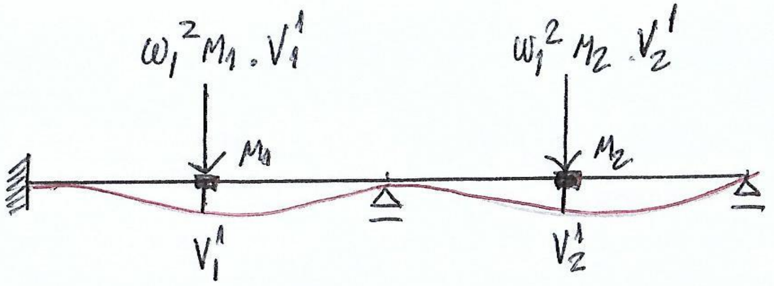
$$W_{ba} = P_{b1} \cdot d_{a1} + P_{b2} \cdot d_{a2} = \underset{\sim}{P_b}^{t} \cdot \underset{\sim}{d_a} = \underset{\sim}{P_b}^{t} \cdot \underline{F} \cdot \underset{\sim}{P_a} \underset{\Uparrow}{=} \underset{\sim}{P_a}^{t} \cdot \underline{F} \cdot \underset{\sim}{P_b} \qquad \text{l.q.q.d.} \checkmark$$

$\underline{F}$ es SIMÉTRICA

16.5 ORTOGONALIDAD DE LOS MODOS DE VIBRACIÓN RESPECTO DE $\boxed{\underline{M}}$

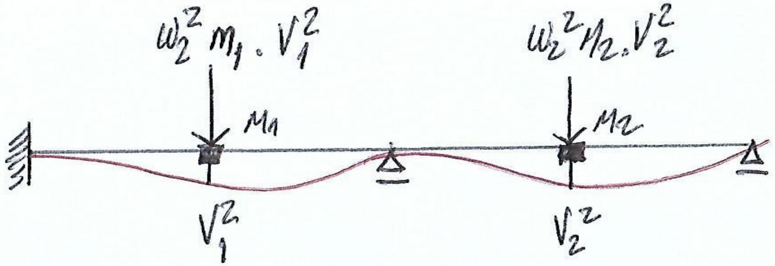
$$\underline{R} \cdot \underset{\sim}{V}^{i} = \omega_i^2 \, \underline{M} \cdot \underset{\sim}{V}^{i}$$

(a)



$$\underset{\sim}{P_a} = \omega_1^2 \, \underline{M} \, \underset{\sim}{V}^1 \qquad \underset{\sim}{d_a} = \underset{\sim}{V}^1 \quad 1^{er} \text{ Modo}$$

(b)



$$\underset{\sim}{P_b} = \omega_2^2 \, \underline{M} \, \underset{\sim}{V}^2 \qquad \underset{\sim}{d_b} = \underset{\sim}{V}^2 \quad 2^{o} \text{ Modo}$$

$$W_{ab} = W_{ba} \Rightarrow \left[\omega_1^2 \, \underline{M} \, \underset{\sim}{V}^1\right]^t \cdot \underset{\sim}{V}^2 = \left[\omega_2^2 \, \underline{M} \cdot \underset{\sim}{V}^2\right]^t \cdot \underset{\sim}{V}^1$$

$$\omega_1^2 \; \underset{\sim}{V}^{1t} \cdot \underline{M} \cdot \underset{\sim}{V}^2 = \omega_2^2 \; \underset{\sim}{V}^{2t} \cdot \underline{M} \cdot \underset{\sim}{V}^1$$

$\underline{M}$ simétrica $\Rightarrow \underset{\sim}{V}^{2t} \cdot \underline{M} \cdot \underset{\sim}{V}^1 = \underset{\sim}{V}^{1t} \cdot \underline{M} \cdot \underset{\sim}{V}^2 \Rightarrow \left(\omega_1^2 - \omega_2^2\right) \cdot \underset{\sim}{V}^{1t} \cdot \underline{M} \cdot \underset{\sim}{V}^2 = 0$

$\omega_1 \neq \omega_2 \Rightarrow \boxed{\underset{\sim}{V}^{1t} \cdot \underline{M} \; \underset{\sim}{V}_2^2 = 0}$

16.6 ORTOGONALIDAD DE LOS MODOS DE VIBRACIÓN RESPECTO DE $\boxed{\underline{R}}$

$$\underline{R} \cdot \underset{\sim}{V}^2 = \omega_2^2\, \underline{M} \cdot \underset{\sim}{V}^2 \Rightarrow -\omega_2^2\, \underline{M} \cdot \underset{\sim}{V}^2 + \underline{R} \cdot \underset{\sim}{V}^2 = \underset{\sim}{0}$$

Premultiplico miembro a miembro por $\underset{\sim}{V}^{1t}$:

$$-\omega_2^2 \cdot \underbrace{\underset{\sim}{V}^{1t} \cdot \underline{M} \cdot \underset{\sim}{V}^2}_{=0} + \underset{\sim}{V}^{1t} \cdot \underline{R} \cdot \underset{\sim}{V}^2 = 0$$

$$\Rightarrow \boxed{\underset{\sim}{V}^{1t} \cdot \underline{R} \cdot \underset{\sim}{V}^2 = 0}$$

En resumen:

$$\boxed{\begin{array}{ll} \forall i \neq k & \underset{\sim}{V}^{it}\, \underline{M}\, \underset{\sim}{V}^k = 0 \\ & \underset{\sim}{V}^{it}\, \underline{R}\, \underset{\sim}{V}^k = 0 \end{array}}$$

16.7

TRANSFORMACIÓN DE UN PROBLEMA DE "n" GRADOS DE LIBERTAD EN "n" PROBLEMAS DE 1 GRADO DE LIBERTAD

$$\underline{M}\,\ddot{\underset{\sim}{X}}(t) + \underline{R}\cdot\underset{\sim}{X}(t) = \underset{\sim}{P}(t)$$

Transformación lineal: $\underset{\sim}{X}(t) = \underline{T}\cdot\underset{\sim}{x}^*(t)$ $\qquad$ $\underline{T}$: Matriz de Transformación

$$\underline{M}\cdot\underline{T}\cdot\ddot{\underset{\sim}{x}}^*(t) + \underline{R}\cdot\underline{T}\cdot\underset{\sim}{x}^*(t) = \underset{\sim}{P}(t)$$

Premultiplico m. a m. por $\underline{T}^t$:

$$\underbrace{\underline{T}^t\cdot\underline{M}\cdot\underline{T}}\cdot\ddot{\underset{\sim}{x}}^*(t) + \underbrace{\underline{T}^t\,\underline{R}\cdot\underline{T}}\cdot\underset{\sim}{x}^*(t) = \underbrace{\underline{T}^t\cdot\underset{\sim}{P}(t)}$$

$$\underline{M}^*\cdot\ddot{\underset{\sim}{x}}^*(t) + \underline{R}^*\cdot\underset{\sim}{x}^*(t) = \underset{\sim}{p}^*(t)$$

Si $\underset{2\times2}{\underline{T}} = \left[\underset{2\times1}{\underset{\sim}{V}^{1}} \;\vdots\; \underset{2\times1}{\underset{\sim}{V}^{2}} \right]$ MATRIZ MODAL

Entonces $\underset{2\times2}{\underline{M}^{*}} = \underset{2\times2}{\begin{bmatrix} \underset{1\times2}{\underset{\sim}{V}^{1t}} \\ \hline \underset{1\times2}{\underset{\sim}{V}^{2t}} \end{bmatrix}} \cdot \underset{2\times2}{\underline{M}} \cdot \left[\underset{2\times1}{\underset{\sim}{V}^{1}} \;\vdots\; \underset{2\times1}{\underset{\sim}{V}^{2}} \right] = \begin{bmatrix} \underset{1\times2}{\underset{\sim}{V}^{1t}} \\ \hline \underset{1\times2}{\underset{\sim}{V}^{2t}} \end{bmatrix} \left[\underbrace{\underset{2\times2}{\underline{M}} \cdot \underset{2\times1}{\underset{\sim}{V}^{1}}}_{2\times1} \;\vdots\; \underbrace{\underset{2\times2}{\underline{M}} \cdot \underset{2\times1}{\underset{\sim}{V}^{2}}}_{2\times1} \right]$

$$\underset{2\times2}{\underline{M}^{*}} = \left[\begin{array}{c|c} \underset{\sim}{V}^{1t} \cdot \underline{M}\, \underset{\sim}{V}^{1} & \underbrace{\underset{\sim}{V}^{1t} \cdot \underline{M}\, \underset{\sim}{V}^{2}}_{=0} \\ \hline \underbrace{\underset{\sim}{V}^{2t}\, \underline{M} \cdot \underset{\sim}{V}^{1}}_{0=} & \underset{\sim}{V}^{2t} \cdot \underline{M} \cdot \underset{\sim}{V}^{2} \end{array} \right] = \left[\begin{array}{c|c} \underset{\sim}{V}^{1t}\underline{M}\,\underset{\sim}{V}^{1} & 0 \\ \hline 0 & \underset{\sim}{V}^{2t}\underline{M}\,\underset{\sim}{V}^{2} \end{array} \right]$$

$\therefore$ $\underline{M}^{*}$ es diagonal $\Rightarrow$ NO HAY ACOPLAMIENTO DE FUERZAS DE INERCIA EN LA NUEVA BASE

$$\underline{R}^* = \begin{bmatrix} \underset{\sim}{V}^{1t} \\ \hline \underset{\sim}{V}^{2t} \end{bmatrix} \cdot \underline{R} \cdot \left[\underset{\sim}{V}^1 \;\middle|\; \underset{\sim}{V}^2 \right] = \begin{bmatrix} \underset{\sim}{V}^{1t} \\ \hline \underset{\sim}{V}^{2t} \end{bmatrix} \left[\underline{R} \cdot \underset{\sim}{V}^1 \;\middle|\; \underline{R} \cdot \underset{\sim}{V}^2 \right]$$

$$\underline{R}^* = \left[\begin{array}{c|c} \underset{\sim}{V}^{1t}\,\underline{R}\,\underset{\sim}{V}^1 & \underbrace{\underset{\sim}{V}^{1t}\,\underline{R}\,\underset{\sim}{V}^2}_{=0} \\ \hline \underbrace{\underset{\sim}{V}^{2t}\,\underline{R}\,\underset{\sim}{V}^1}_{=0} & \underset{\sim}{V}^{2t} \cdot \underline{R} \cdot \underset{\sim}{V}^2 \end{array} \right] = \left[\begin{array}{c|c} \underset{\sim}{V}^{1t} \cdot \underline{R} \cdot \underset{\sim}{V}^1 & 0 \\ \hline 0 & \underset{\sim}{V}^{2t} \cdot \underline{R} \cdot \underset{\sim}{V}^2 \end{array} \right]$$

$\therefore$ $\underline{R}^*$ es diagonal $\Rightarrow$ NO HAY ACOPLAMIENTO DE FUERZAS ELÁSTICAS

$$\text{Si} \quad \boxed{\underline{T} = \left[\frac{\underset{\sim}{V}^1}{\sqrt{\underset{\sim}{V}^{1t}\,\underline{M}\,\underset{\sim}{V}^1}} \;\middle|\; \frac{\underset{\sim}{V}^2}{\sqrt{\underset{\sim}{V}^{2t}\,\underline{M}\,\underset{\sim}{V}^2}} \right]}$$

MATRIZ MODAL EN COORDENADAS NORMALES

$$\underline{M}^* = \left[\begin{array}{c|c} \frac{\underset{\sim}{V}^{1t}}{\sqrt{\underset{\sim}{V}^{1t}\underline{M}\,\underset{\sim}{V}^1}}\;\underline{M}\cdot\frac{\underset{\sim}{V}^1}{\sqrt{\underset{\sim}{V}^{1t}\underline{M}\,\underset{\sim}{V}^1}} & 0 \\ \hline 0 & \frac{\underset{\sim}{V}^{2t}}{\sqrt{\underset{\sim}{V}^{2t}\underline{M}\,\underset{\sim}{V}^2}}\;\underline{M}\;\frac{\underset{\sim}{V}^2}{\sqrt{\underset{\sim}{V}^{2t}\underline{M}\,\underset{\sim}{V}^2}} \end{array} \right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \right]$$

$$\boxed{\underline{M}^* = \underline{I}}$$

¡ $\underline{M}^*$ es la MATRIZ UNIDAD !

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$$\underline{R}^* = \left[\begin{array}{c|c} \dfrac{\underset{\sim}{V}^{1t}\,\underline{R}\cdot\underset{\sim}{V}^1}{\underset{\sim}{V}^{1t}\cdot\underline{M}\cdot\underset{\sim}{V}^1} & 0 \\ \hline 0 & \dfrac{\underset{\sim}{V}^{2t}\,\underline{R}\,\underset{\sim}{V}^2}{\underset{\sim}{V}^{2t}\cdot\underline{M}\,\underset{\sim}{V}^2} \end{array}\right] = \left[\begin{array}{c|c} \omega_1^2 & 0 \\ \hline 0 & \omega_2^2 \end{array}\right] = \underline{\lambda}$$

MATRIZ ESPECTRAL

$$\Uparrow$$

$$\underline{R}\cdot\underset{\sim}{V}^1 = \omega_1^2\,\underline{M}\cdot\underset{\sim}{V}^1$$

Premult. por $\underset{\sim}{V}^{1t}$:

$$\underset{\sim}{V}^{1t}\,\underline{R}\cdot\underset{\sim}{V}^1 = \omega_1^2\,\underset{\sim}{V}^{1t}\,\underline{M}\,\underset{\sim}{V}^1 \Rightarrow \frac{\underset{\sim}{V}^{1t}\,\underline{R}\,\underset{\sim}{V}^1}{\underset{\sim}{V}^{1t}\,\underline{M}\,\underset{\sim}{V}^1} = \omega_1^2$$

¿Cómo queda el sistema de ecuaciones de Equilibrio Dinámico en la nueva Base?

$$\ddot{\underset{\sim}{x}}^{*}(t) + \underline{\lambda} \cdot \underset{\sim}{x}^{*}(t) = \underset{\sim}{p}^{*}(t)$$

ECUACIONES DIFERENCIALES DESACOPLADAS

$$\Downarrow$$

$$\begin{cases} \ddot{x}_1^{*}(t) + \omega_1^2 \, x_1^{*}(t) = p_1^{*}(t) \\ \ddot{x}_2^{*}(t) + \omega_2^2 \cdot x_2^{*}(t) = p_2^{*}(t) \end{cases}$$

Resuelvo cada ecuación en forma independiente y obtengo:

$$x_1^*(t) = \text{- - - -}$$

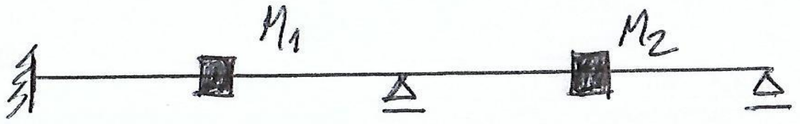
$$x_2^*(t) = \text{——}$$

$$\Rightarrow \underset{\sim}{x}^*(t) = \begin{Bmatrix} x_1^*(t) \\ x_2^*(t) \end{Bmatrix}$$

y finalmente:

$$\underset{\sim}{X}(t) = \underline{T} \cdot \underset{\sim}{x}^*(t)$$

$$\underset{\sim}{V}^1 = \begin{Bmatrix} 1 \\ -2{,}415 \end{Bmatrix} \qquad \underset{\sim}{V}^2 = \begin{Bmatrix} 1 \\ 0{,}414 \end{Bmatrix}$$



$$\underset{\sim}{V}^{1t} \cdot \underline{M} \cdot \underset{\sim}{V}^1 = \begin{bmatrix} 1 & -2{,}415 \end{bmatrix} \begin{bmatrix} 1000 & 0 \\ 0 & 1000 \end{bmatrix} \cdot \begin{Bmatrix} 1 \\ -2{,}415 \end{Bmatrix} = 6832{,}225 \Rightarrow \sqrt{} = 82{,}657$$

$$\underset{\sim}{V}^{2t} \cdot \underline{M} \cdot \underset{\sim}{V}^2 = \begin{bmatrix} 1 & 0{,}414 \end{bmatrix} \begin{bmatrix} 1000 & 0 \\ 0 & 1000 \end{bmatrix} \cdot \begin{Bmatrix} 1 \\ 0{,}414 \end{Bmatrix} = 1171{,}396 \Rightarrow \sqrt{} = 34{,}226$$

$$\underline{T} = \left[\frac{\underset{\sim}{V}^1}{\sqrt{\underset{\sim}{V}^{1t}\underline{M}\underset{\sim}{V}^1}} \;\middle|\; \frac{\underset{\sim}{V}^2}{\sqrt{\underset{\sim}{V}^{2t}\underline{M}\underset{\sim}{V}^2}} \right] = \begin{bmatrix} 0{,}012 & 0{,}029 \\ -0{,}029 & 0{,}012 \end{bmatrix}$$

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